NSOP$_1$ theories

Symmetry

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ASL North American Annual Meeting
July 21, 2026

nickramsey621.github.io/nsop1-tutorial.html

Outline

1. The evolution of independence
2. Trees, Kim's Lemma, and the generic scale
3. **Symmetry**

Conventions

- We work always in a complete theory T in a langauge L with monster model $\mathbb{M}$, which is assumed to be very saturated and strongly homogeneous.
- Unless mentioned otherwise, all sequences, tuples, models, etc. are assumed to be living in $\mathbb{M}$.
- We refer to complete types over $\mathbb{M}$ as *global* types.
- We write $a \equiv_C b$ to indicate $\mathrm{tp}(a/C) = \mathrm{tp}(b/C)$.
- We write $L(A)$ for formulas in L with parameters from A.

Last time

- We introduced global M-invariant types: these are types with the property that if $\varphi(x; y) \in L(M)$ and $a \equiv_M b$, then $\varphi(x; a) \in q$ if and only if $\varphi(x; b) \in q$—in other words, membership of $\varphi(x; c) \in q$ depends only on $\mathrm{tp}(c/M)$.
- We showed that every $p \in S(M)$ extends to a global M-invariant type. The construction was to pick an ultrafilter $\mathcal{D}$ on M extending the filter generated by $\{\varphi(M) : \varphi \in p\}$ and considering

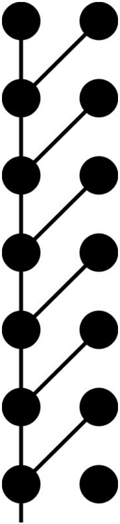
$$\mathrm{Av}(\mathcal{D}, \mathbb{M}) = \{\psi(x) \in L(\mathbb{M}) : \psi(M) \in \mathcal{D}\}.$$

- We also described $\otimes$ on global M-invariant types, where $(a, b) \models (p \otimes q)|_A$ if and only if $b \models q$ and $a \models p|_{Ab}$.

Last time

- We also saw that $\varphi(x; y)$ has SOP_1 if and only if there is an array $(c_{i,j})_{i<\omega, j<2}$ such that
  1. $\{\varphi(x; c_{i,0}) : i < \omega\}$ is consistent.
  2. $\{\varphi(x; c_{i,1}) : i < \omega\}$ is 2-inconsistent.
  3. For all i, $c_{i,0} \equiv_{\overline{c}_{<i}} c_{i,1}$.
- We may replace 2 with $k > 2$ but then the formula witnessing SOP_1 will change.
- By Ramsey's theorem and compactness, we may assume $(\overline{c}_i)_{i<\omega}$ is an indiscernible sequence.

The array



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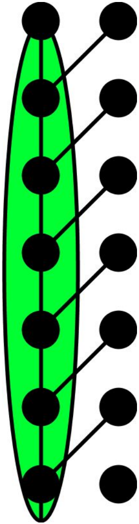

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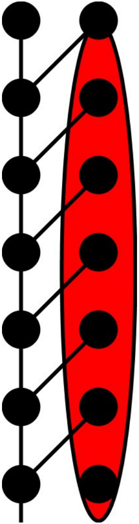

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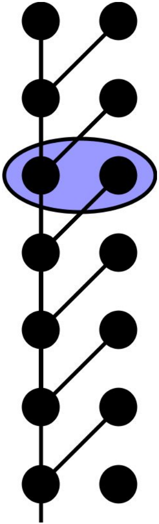

Kim's Lemma for Kim-dividing

Recall that we say $\varphi(x; a)$ *Kim-divides* over $M \models T$ if, for some Morley sequence over M $(a_i)_{i<\omega}$ in a global M-invariant type, with $a_0 = a$, $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent.

Theorem

(Kaplan-R.) The following are equivalent:

1. *T is $NSOP_1$*
2. *Kim's Lemma for Kim-dividing: for any model $M \models T$, $\varphi(x; a)$ Kim-divides over M if and only if $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent for every Morley sequence $(a_i)_{i<\omega}$ in a global M-invariant type extending $\mathrm{tp}(a/M)$.*

Array implies failure of Kim's Lemma

- Now suppose we have the array $(c_{i,0}, c_{i,1})_{i<\omega}$ satisfying
  1. $\{\varphi(x; c_{i,0}) : i < \omega\}$ is consistent.
  2. $\{\varphi(x; c_{i,1}) : i < \omega\}$ is 2-inconsistent.
  3. $c_{i,0} \equiv_{\overline{c}_{<i}} c_{i,1}$.
- We may assume that we are working in a Skolemization T^{Sk} of T, that we have stretched the array to $(c_{i,0}, c_{i,1})_{i<\omega+\omega}$, and that $(c_{i,0}, c_{1,1})_{i<\omega+1}$ is an indiscernible sequence.
- Let $M = \mathrm{Sk}(\overline{c}_{<\omega})$ and let $\mathcal{D}_j$ be a non-principal ultrafilter on M concentrating on $\{c_{i,j} : i < \omega\}$.
- By indiscernibility, we have $c_{\omega,j} \models \mathrm{Av}(\mathcal{D}_j, M)$ and since $c_{\omega,0} \equiv_{\overline{c}_{<\omega}} c_{\omega,1}$ and hence $c_{\omega,0} \equiv_M c_{\omega,1}$, we have $c_{\omega,0} \equiv_M c_{\omega,1}$.
- But, $(c_{\omega+i,j})$ is Morley over M in $\mathrm{Av}(\mathcal{D}_j, \mathbb{M})$ for $j = 0, 1$ (indexed in reverse), $\{\varphi(x; c_{\omega+i,0}) : i < \omega\}$ is consistent, and $\{\varphi(x; c_{\omega+1,1}) : i < \omega\}$ is inconsistent. This is a failure of Kim's Lemma for Kim-dividing.

Failure of Kim's Lemma implies array

- Suppose Kim's Lemma fails: so we have $\varphi(x; a)$ Kim-divides over M, witnessed by a Morley sequence $(a_i)_{i<\omega}$ in some global M-invariant type $r \supseteq \mathrm{tp}(a/M)$ and there is some global M-invariant type $q \supseteq \mathrm{tp}(a/M)$ such that, whenever $(a_i)_{i<\omega}$ is a Morley sequence over M in q, $\{\varphi(x; a_i) : i < \omega\}$ is consistent.
- Let $(c_{i,0}, c_{i,1})_{i\in\mathbb{Z}} \models (q \otimes r)^{\otimes\mathbb{Z}}|_M$.
- By choice of q and r, we have $\{\varphi(x; c_{i,0}) : i \in \mathbb{Z}\}$ consistent and $\{\varphi(x; c_{i,1}) : i \in \mathbb{Z}\}$ is k-inconsistent for some k.
- Moreover, for each i, $c_{i,0} \equiv_M c_{i,1}$ (since both have the same type as a over M) and

  $$(c_{>i,0}, c_{>i,1}) \models (q \otimes r)^{\otimes\omega}|_{Mc_{i,0},c_{i,1}}$$

  so, by M-invariance, $c_{i,0} \equiv_{M\overline{c}_{>i}} c_{i,1}$.
- Then $(c_{-i,0}, c_{-i,1})_{i<\omega}$ is the array we want.

Kim's Lemma for Kim-dividing

Theorem

(Kaplan-R.) The following are equivalent:

1. *T is NSOP$_1$*
2. *Kim's Lemma for Kim-dividing: for any model $M \models T$, $\varphi(x; a)$ Kim-divides over M if and only if $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent for every Morley sequence $(a_i)_{i<\omega}$ in a global M-invariant type extending $\mathrm{tp}(a/M)$.*

Consequences of Kim's Lemma

- Last time, we saw that, in the context of simple theories, Kim's Lemma had two important consequences:
  1. Forking = dividing
  2. Symmetry
- An identical argument shows that, if T is NSOP_1, then Kim-forking = Kim-dividing.
- What about symmetry?

One more time: symmetry of $\mathop{\smile\!\!\!\!\!\!\!\;|}\,^f$ in simple theories

1. Working in a simple theory, suppose $a \mathop{\smile\!\!\!\!\!\!\!\;|}^f_A b$ and we will show $b \mathop{\smile\!\!\!\!\!\!\!\;|}^f_A a$.

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2. Let $a_0 = a$ and use transfinite recursion and extension to choose a_α so that $a_\alpha \models \mathrm{tp}(a/Ab)$ and $a_\alpha \mathop{\smile\!\!\!\!\!\!|}^f_A a_{<\alpha} b$ for all α in some large enough κ. By Erdős-Rado, we may assume $(a_\alpha)_{\alpha \in \kappa}$ is Ab-indiscernible.

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3. Now suppose $\varphi(x; a) \in \mathrm{tp}(b/Aa)$. Since $a_0 = a$ and $(a_\alpha)_{\alpha\in\kappa}$ is Ab-indiscernible, we have $b \models \{\varphi(x; a_\alpha) : \alpha \in \kappa\}$.

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4. But since $a_\alpha \mathop{\smile\!\!\!\!\!\!\,|}^f_A a_{<\alpha}$, the sequence $(a_\alpha)_{\alpha \in \kappa}$ is a Morley sequence over A so (3) entails $\varphi(x; a)$ does not divide (and therefore does not fork) over A.

One more time: symmetry of $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^f$ in simple theories

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2. Let $a_0 = a$ and use transfinite recursion and extension to choose a_α so that $a_\alpha \models \mathrm{tp}(a/Ab)$ and $a_\alpha \mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^f_A a_{<\alpha} b$ for all α in some large enough κ. By Erdős-Rado, we may assume $(a_\alpha)_{\alpha\in\kappa}$ is Ab-indiscernible.
3. Now suppose $\varphi(x; a) \in \mathrm{tp}(b/Aa)$. Since $a_0 = a$ and $(a_\alpha)_{\alpha\in\kappa}$ is Ab-indiscernible, we have $b \models \{\varphi(x; a_\alpha) : \alpha \in \kappa\}$.
4. But since $a_\alpha \mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^f_A a_{<\alpha}$, the sequence $(a_\alpha)_{\alpha\in\kappa}$ is a Morley sequence over A so (3) entails $\varphi(x; a)$ does not divide (and therefore does not fork) over A.
5. But $\varphi(x; a) \in \mathrm{tp}(b/Aa)$ was arbitrary, so $b \mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^f_A a$.

One more time: symmetry of $\mathop{\smile\hskip-0.9em\vert\hskip0.6em}^{f}$ in simple theories

- A distillation of the argument:
  1. (Existence) The hypothesis $a \mathop{\smile\hskip-0.9em\vert\hskip0.6em}^{f}_{A} b$ entails the existence of a Morley sequence $(a_i)_{i<\omega}$ which is Ab-indiscernible.
  2. (Witnessing) Morley sequences witness dividing (by Kim's Lemma) so the Ab-indiscernibility of $(a_i)_{i<\omega}$ implies $b \mathop{\smile\hskip-0.9em\vert\hskip0.6em}^{f}_{A} a$.
- We have a difficulty in generalizing this to the NSOP_1 setting:
  1. The $a \mathop{\smile\hskip-0.9em\vert\hskip0.6em}^{K}_{M} b$ guarantees the existince of an $\mathop{\smile\hskip-0.9em\vert\hskip0.6em}^{K}$-Morley sequence which is Ab-indiscernible, but $\mathop{\smile\hskip-0.9em\vert\hskip0.6em}^{K}$-Morley is much weaker than being Morley in a global M-invariant type. We do not know that such sequences witness Kim-dividing.
  2. If $\mathrm{tp}(a/Mb)$ extends to a global M-invariant type q, then we get an Mb-indiscernible sequence $(a_i)_{i<\omega}$ which is Morley in a global M-invariant type. But this hypothesis is *much* stronger than $a \mathop{\smile\hskip-0.9em\vert\hskip0.6em}^{K}_{M} b$.
- We need anotion of 'generic sequence' which is weak enough to exist under the hypothesis $a \mathop{\smile\hskip-0.9em\vert\hskip0.6em}^{K}_{M} b$ and yet also strong enough to witness Kim-dividing, so we can conclude $b \mathop{\smile\hskip-0.9em\vert\hskip0.6em}^{K}_{M} a$.

Generalized indiscernibles

- Suppose I is an L'-structure. We say $(a_i)_{i\in I}$ is *I-indexed indiscernible* if whenever $\bar{i}$ and $\bar{j}$ are tuples from I with $\mathrm{qftp}_{L'}(\bar{i}) = \mathrm{qftp}_{L'}(\bar{j})$, then $a_{\bar{i}} \equiv a_{\bar{j}}$ (if $\bar{i} = (i_0, \ldots, i_{n-1})$, we write $a_{\bar{i}}$ for $(a_{i_0}, \ldots, a_{i_{n-1}})$.
- This generalizes the usual notion of indiscernible sequence: an indiscernible sequence is an I-indexed indiscernible where I is an infinite linear order in the language $L' = \{\leq\}$.
- We say $(b_i)_{i\in I}$ is *locally based* on $(a_i)_{i\in I}$ if, whenever $\varphi \in L$, if $\models \varphi(b_{\bar{i}})$, then there is some $\bar{j}$ in I with $\mathrm{qftp}(\bar{j}) = \mathrm{qftp}(\bar{i})$ such that $\models \varphi(a_{\bar{j}})$.
- Same but different: define $\mathrm{EM}_I(a_i : i \in I)$ to be the partial type in the variables $(x_i)_{i\in I}$ with $\varphi(x_{\bar{i}}) \in \mathrm{EM}_I(a_i : i \in I)$ if and only if $\models \varphi(a_{\bar{j}})$ for all $\bar{j} \models \mathrm{qftp}(\bar{i})$ in I. Then $(b_i)_{i\in I}$ is locally based on $(a_i)_{i\in I}$ if and only if $(b_i)_{i\in I}$ realizes $\mathrm{EM}_I(a_i : i \in I)$.

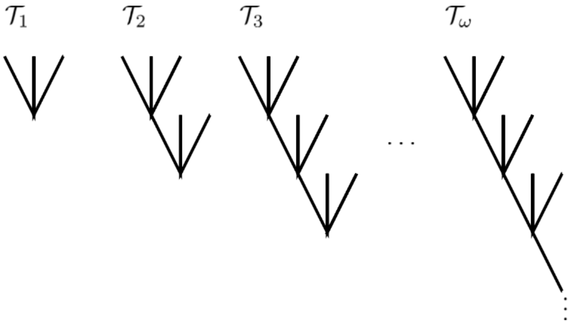
Generalized indiscernibles

- We say I-indexed indiscernibles have the *modeling property* if, given any $(a_i)_{i \in I}$, there is some I-indexed indiscernible $(b_i)_{i \in I}$ locally based on $(a_i)_{i \in I}$.
- **Scow's Theorem**: I-indexed indiscernibles have the modeling property if and only if $\mathrm{Age}(I)$ is a Ramsey class.
- When $I = \omega^{<\omega}$ is a structure in the language $L_s = \{\trianglelefteq, \leq_{lex}, \wedge, (P_\alpha)_{\alpha<\omega}\}$, where the predicate P_n names nodes of length n, we refer to I-indexed indiscernibles as s-indiscernible trees. It is known that s-indiscernible trees have the modeling property.
- We will also refer to indiscernibles indexed by trees in this language but with a different number of levels also as s-indiscernible trees.

A class of trees

- Let α be an ordinal. We define $\mathcal{T}_\alpha$ to be the set of functions f with the following properties:
  - $\mathrm{dom}(f) = [\beta, \alpha)$ for some non-limit $\beta \leq \alpha$.
  - $\mathrm{ran}(f) = \omega$.
  - (finite support) $\{\gamma \in \mathrm{dom}(f) : f(\gamma) \neq 0\}$ is finite.
- (Canonical inclusions) For $\alpha < \beta$, $\iota_{\alpha,\beta}(f) = f \cup \{(\gamma, 0) : \gamma \in \beta \setminus \alpha\}$. When δ is limit, $\mathcal{T}_\delta = \varinjlim_{\alpha<\delta} \mathcal{T}_\alpha$.
- (Restriction) If $w \subseteq \alpha$, we define $\mathcal{T}_\alpha \restriction w$ to be the set of $\eta \in \mathcal{T}_\alpha$ such that $\min(\mathrm{dom}(\eta)) \in w$ and $\beta \in \mathrm{dom}(\eta) \setminus w \implies \eta(\beta) = 0$. Note that $\mathcal{T}_\alpha \restriction w \cong \mathcal{T}_{\mathrm{ot}(w)}$.
- (Concatenation) If $\eta \in \mathcal{T}_\alpha$ and $i < \omega$, we write $\langle i \rangle^\frown \eta = \eta \cup \{(\alpha, i)\} \in \mathcal{T}_{\alpha+1}$.

A class of trees



Morley trees

- We say $(a_\eta)_{\eta \in \mathcal{T}_\alpha}$ is *spread out* over M if, for all $\eta \in \mathcal{T}_\alpha$ with $\mathrm{dom}(\eta) \neq \alpha$, $\langle a_{\trianglerighteq \eta^\frown \langle i \rangle} : i < \omega \rangle$ is a Morley sequence in a global M-invariant type.
- We say $(a_\eta)_{\eta \in \mathcal{T}_\alpha}$ is a *Morley tree* over M if $\alpha \geq \omega$ and
  - $(a_\eta)_{\eta \in \mathcal{T}_\alpha}$ is s-indiscernible and spread out over M.
  - For all finite $w, w' \subseteq \alpha$ with $|w| = |w'|$, $(a_\eta)_{\eta \in \mathcal{T}_\alpha \restriction w} \equiv_M (a_\eta)_{\eta \in \mathcal{T}_\alpha \restriction w'}$.
- A *tree Morley sequence* over M is a sequence that is (or, equivalently, has the same type as) the all zeros path in some Morley tree.

Tree Morley sequences exist

- If $a \mathop{\smile\!\!\!\!\!\!\,|}^K b$, there is a tree Morley sequence over M $(a_i)_{i<\omega}$ with $a_0 = a$ which is Mb-indiscernible.
- The proof goes by first constructing a very tall tree which is s-indiscernible and spread out over M, and then applies Erdős-Rado.

The tree construction

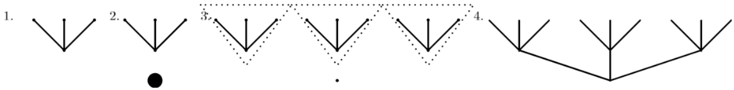


Figure 2. The figure displays the construction of the tree indexed by $\mathcal{T}_2$ in stages: 1. The tree indexed by $\mathcal{T}_1$. 2. Using extension to obtain a new base point. 3. Taking a Morley sequence in the given tree, indiscernible over the new base point. 4. Extracting an s-indiscernible tree to obtain a spread out, s-indiscernible tree indexed by $\mathcal{T}_2$.

Witnessing for tree Morley sequences

- Suppose T is NSOP$_1$ and $\varphi(x; a)$ Kim-divides over M.
- Let $(a_i)_{i<\omega}$ be a tree Morley sequence over M with $a_0 = a$. We must show that $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent.
- Let $\zeta_i \in \mathcal{T}_\omega$ be the constantly zero function with domain $[i, \omega)$ and let $\eta_i = \{(i, 1)\} \cup \zeta_{i+1}$. Let $(a_\eta)_{\eta \in \mathcal{T}_\omega}$ be a Morley tree over M with $a_i = a_{\zeta_i}$.
- By spreadoutness, $(a_{\eta_i})_{i<\omega}$ is a Morley sequence in a global M-invariant type, so, by Kim's Lemma for Kim-dividing, $\{\varphi(x; a_{\eta_i}) : i < \omega\}$ is inconsistent.
- By s-indiscernibility, $a_{\zeta_i} \equiv_{Ma_{\zeta_{>i}} a_{\nu_{>i}}} a_{\eta_i}$.
- Thus, if $\{\varphi(x; a_{\zeta_i}) : i < \omega\}$ is consistent, we obtain the SOP$_1$ array.

Symmetry

Corollary

If T is $NSOP_1$, $\mathop{\smile\!\!\!\!\!\!\,|}^K$ is symmetric.

Thanks!