NSOP$_1$ theories

Trees, Kim's Lemma, and the Generic Scale

Nicholas Ramsey

University of Notre Dame

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nickramsey621.github.io/nsop1-tutorial.html

Outline

1. The evolution of independence
2. **Trees, Kim's Lemma, and the generic scale**
3. Symmetry

Conventions

- We work always in a complete theory T in a langauge L with monster model $\mathbb{M}$, which is assumed to be very saturated and strongly homogeneous.
- Unless mentioned otherwise, all sequences, tuples, models, etc. are assumed to be living in $\mathbb{M}$.
- We refer to complete types over $\mathbb{M}$ as *global* types.
- We write $a \equiv_C b$ to indicate $\mathrm{tp}(a/C) = \mathrm{tp}(b/C)$.
- We write $L(A)$ for formulas in L with parameters from A.

Forking

- We say a formula $\varphi(x; a)$ *divides* over A if there are are infinitely many realizations $a_i \models \mathrm{tp}(a/A)$ and some k such that $\{\varphi(x; a_i) : i < \omega\}$ is k-inconsistent. One can demand demand that $(a_i)_{i<\omega}$ forms an A-indiscernible sequence and this is equivalent.
- A formula $\varphi(x; a)$ *forks* over A if

$$\varphi(x; a) \vdash \bigvee_{i<\ell} \psi_i(x; c_i)$$

where each $\psi_i(x; c_i)$ divides over A.
- We write $a \mathop{\smile\!\!\!\!\!\mid}^{f}_{A} b$ to indicate that $\mathrm{tp}(a/Ab)$ does not fork over A.

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- Try a compactness argument to show that

$$p(x) \cup \{\neg\varphi(x;c) : \varphi(x;c) \text{ forks over } A, c \in C\}$$

is consistent. If not, some formula in p implies a finite disjunction of formulas that fork over A, so p forks over A.

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  is consistent. If not, some formula in p implies a finite disjunction of formulas that fork over A, so p forks over A.
- Thus, we are forced to consider finite disjunctions.

Morley sequences

- Dividing is defined by an *existential* condition: $\varphi(x; a)$ divides A if *there exists* some A-indiscernible sequence with $a_0 = a$ such that $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent.

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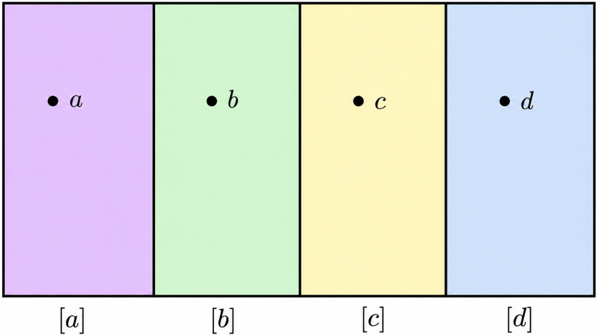
- Which sequences witness the inconsistency? Obviously not all (e.g. the constant sequence is A-indiscernible but doesn't witness inconsistency unless $\varphi(x; a)$ is the false formula).
- We say a sequence $(a_i)_{i<\omega}$ is a *Morley sequence* over A if it is A-indiscernible and $a_i \mathop{\smile\!\!\!\!\!\vert}^f_A a_{<i}$ for all i.

Morley sequences



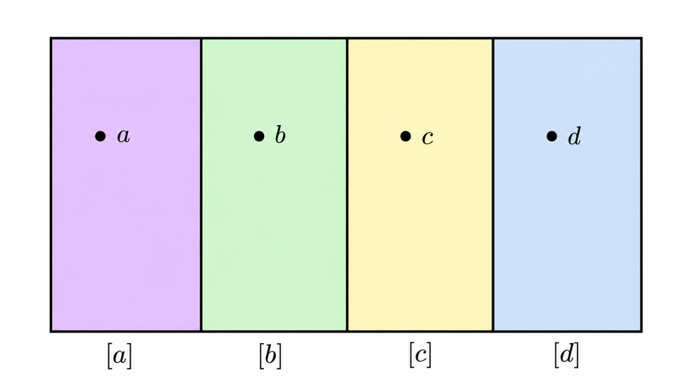
Kim's Lemma

Theorem

Suppose T is a simple theory. Then the following are equivalent:

1. *$\varphi(x; a)$ divides over A.*
2. *There is a Morley sequence $(a_i)_{i<\omega}$ over A with $a_0 = a$ such that $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent.*
3. *For every Morley sequence $(a_i)_{i<\omega}$ over A with $a_0 = a$, $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent.*

Morley sequences



$E(x, a)$ only divides along the Morley sequence

Consequence 1 of Kim's Lemma: Forking = Dividing

1. Let $\varphi(x; a) \vdash \bigvee_{j<\ell} \psi_j(x; b_j)$ where each $\psi_j(x; b_j)$ divides over A. We want to show that $\varphi(x; a)$ divides over A as well.

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2. To this end, we take a Morley sequence $(a_i, b_{i,0}, \ldots, b_{i,\ell-1})_{i<\omega}$ over A starting with $(a, b_0, \ldots, b_{\ell-1})$. Then $(a_i)_{i<\omega}$ and each $(b_{i,j})_{i<\omega}$ are Morley sequences over A.

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4. Indiscernibility entails that $\varphi(x; a_i) \vdash \bigvee_{j<\ell} \psi_j(x; b_{i,j})$ for all i so, by the pigeonhole principle, there must be some infinite set X and some j_* such that c realizes $\{\psi_{j_*}(x; b_{i,j_*}) : i \in X\}$.

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5. Indiscernibility again entails that $\{\psi_{j_*}(x; b_{i,j_*}) : i < \omega\}$ is consistent. Contradicts Kim's Lemma, since we assumed that $\psi_{j_*}(x; b_{j_*})$ divides over A. $\square$

Consequence 2 of Kim's Lemma: Symmetry

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3. Now suppose $\varphi(x; a) \in \mathrm{tp}(b/Aa)$. Since $a_0 = a$ and $(a_\alpha)_{\alpha\in\kappa}$ is Ab-indiscernible, we have $b \models \{\varphi(x; a_\alpha) : \alpha \in \kappa\}$.

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4. But since $a_\alpha \mathop{\smile\!\!\!\!\!\!\:|}^{f}_{A} a_{<\alpha}$, the sequence $(a_\alpha)_{\alpha \in \kappa}$ is a Morley sequence over A so (3) entails $\varphi(x; a)$ does not divide (and therefore does not fork) over A.

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5. But $\varphi(x; a) \in \mathrm{tp}(b/Aa)$ was arbitrary, so $b \mathop{\smile\!\!\!\!\!\!\,|}_A^f a$.

Problem 1: Kim's Lemma fails outside of simple theories

- Recall the definition of T^*_{feq} from last time: the language consists of two sorts P and O and a ternary relation $E_x(y, z) \subseteq P \times O^2$. The theory T_{feq} says that, for each $p \in P$, E_p is an equivalence relation on O and T^*_{feq} is the model companion.

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- Consider the formulas $E_p(x,a)$, where $p \in P$ and $a \in O$. This divides over $\emptyset$. Choose an indiscernible sequence $\langle (a_i, p) : i < \omega \rangle$ such that $\models \neg E_p(a_i, a_j)$ for all $i \neq j$.
- But $E_p(x,a)$ does *not* divide along any Morley sequence over $\emptyset$: if $\langle a_i, p_i : i < \omega \rangle$ is a Morley sequence over $\emptyset$, then $p_i \neq p_j$ when $i \neq j$ and then $\{E_{p_i}(x, a_i) : i < \omega\}$ is consistent.

Problem 2: Types can fork over their own domain

- Consider the structure $M = (\omega, \mathcal{P}(\omega), \in)$ where ω and $\mathcal{P}(\omega)$ live on separate sorts and $\in$ is the elementhood relation.
- Note that each permutation $\sigma \in \mathrm{Sym}(\omega)$ induces an automorphism of M. In particular, if $X, Y \in \mathcal{P}(\omega)$ are both infinite and co-infinite sets, then $\mathrm{tp}(X) = \mathrm{tp}(Y)$.
- If x is a variable in the ω sort, we have

$$x = x \vdash x \in \mathrm{Evens} \vee x \in \mathrm{Odds}.$$

- Let $(Z_i)_{i<\omega}$ be a sequence of pairwise disjoint infinite co-infinite elements of $\mathcal{P}(\omega)$. Then $\mathrm{tp}(Z_i) = \mathrm{tp}(\mathrm{Evens}) = \mathrm{tp}(\mathrm{Odds})$ and

$$\{x \in Z_i : i < \omega\}$$

is 2-inconsistent (for example, take $Z_i = \{p_i^n : n \geq 1\}$ for p_i the ith prime).
- This shows both $x \in \mathrm{Evens}$ and $x \in \mathrm{Odds}$ divide, so $x = x$ forks.

Global finitely satisfiable extensions

Let $p \in S(M)$ be a complete type over a model $M \models T$. Then there is a global $q \in S(\mathbb{M})$ extending p such that q is finitely satisfiable in M.

1. Argument for one kind of model-theorist: let

   $$q_0(x) = p(x) \cup \{\neg\psi(x) : \psi(x) \in L(\mathbb{M}) \text{ is not } M\text{-finitely satisfiable}\},$$

   and show this is consistent by compactness. Take q to be any completion.
2. Argument for another kind of model-theorist: the set

   $$\mathcal{F} = \{X \subseteq \mathcal{P}(M) : \varphi(M) \subseteq X \text{ for some } \varphi \in p\}$$

   is a filter on M; let $\mathcal{D}$ be any extension to an ultrafilter. Then consider

   $$\mathrm{Av}(\mathcal{D}, \mathbb{M}) := \{\psi(x) \in L(\mathbb{M}) : \psi(M) \in \mathcal{D}\}.$$

   Then $\mathrm{Av}(\mathcal{D}, \mathbb{M})$ extends p and is M-finitely satisfiable.

Invariant types

- A global M-finitely satisfiable type q has a key property called *M-invariance*: if $\varphi(x; y) \in L(M)$ and $a \equiv_M b$, then $\varphi(x; a) \in q$ if and only if $\varphi(x; b) \in q$—in other words, membership of $\varphi(x; c) \in q$ depends only on $\mathrm{tp}(c/M)$. [To see this, suppose not and consider $\varphi(x; a) \wedge \neg\varphi(x; b) \in q$; a realization in M contradicts $a \equiv_M b$].
- There is an operation $\otimes$ on global M-invariant types so that $(a, b) \models (p \otimes q)|_A$ if and only if $b \models q|_A$ and $a \models p|_{Ab}$. The global type $p \otimes q$ is again M-invariant. For average types, this agrees with the usual Fubini product of ultrafilters: $\mathrm{Av}(\mathcal{D}, \mathbb{M}) \otimes \mathrm{Av}(\mathcal{E}, \mathbb{M}) = \mathrm{Av}(\mathcal{D} \otimes \mathcal{E}, \mathbb{M})$.
- Can continue: we write $q^{\otimes\omega}$ for the global type such that $(a_i)_{i<\omega} \models q^{\otimes\omega}|_A$ if and only if $a_i \models q|_{Aa_{<i}}$ for all i. If $(a_i)_{i<\omega} \models q^{\otimes\omega}|_M$ then $(a_i)_{i<\omega}$ is automatically M-indiscernible and $a_i \mathop{\smile\!\!\!\!\!\!|}\,^f_M a_{<i}$ so $(a_i)_{i<\omega}$ is a Morley sequence over M.
- Upshot: Morley sequences in types over models always exist.

Kim-dividing

Definition

Let $M \models T$.

1. Say $\varphi(x; a)$ *Kim-divides* over M if there is a Morley sequence $(a_i)_{i<\omega} \models q^{\otimes\omega}|_M$ over M, in some global M-invariant type $q \supseteq \mathrm{tp}(a/M)$, such that $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent.
2. Say $\varphi(x; a)$ *Kim-forks* over M if $\varphi(x; a) \vdash \bigvee_{j<\ell} \psi_j(x; c_j)$ where each $\psi_j(x; c_j)$ Kim-divides over M.
3. Write $a \mathop{\smile\!\!\!\!\!\!\;|}\,^K_M b$ to mean that $\mathrm{tp}(a/Mb)$ contains no formulas that Kim-fork over M.

Slogan: Kim-independence is *independence at the generic scale*

Kim's Lemma for Kim-dividing

Theorem

(Kaplan-R.) The following are equivalent:

1. *T is NSOP$_1$*
2. *Kim's Lemma for Kim-dividing: for any model $M \models T$, $\varphi(x; a)$ Kim-divides over M if and only if $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent for every Morley sequence $(a_i)_{i<\omega}$ in a global M-invariant type extending $\mathrm{tp}(a/M)$.*

Reminder: NSOP_1 theories

The following definition is due to Džamonja and Shelah.

Definition

We say a formula $\varphi(x; y)$ has SOP_1 if there exists a collection of tuples $(a_\eta)_{\eta \in 2^{<\omega}}$ satisfying the following conditions:

1. (Paths are consistent) For all $\eta \in 2^\omega$, $\{\varphi(x; a_{\eta|n}) : n < \omega\}$ is consistent.

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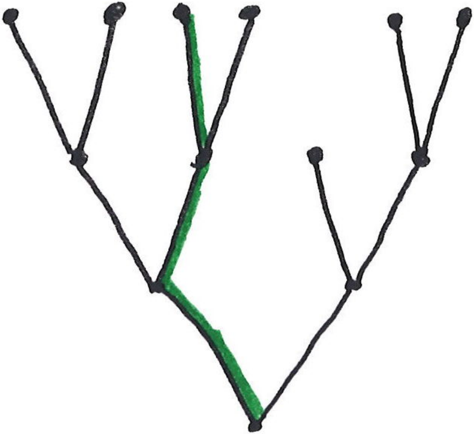
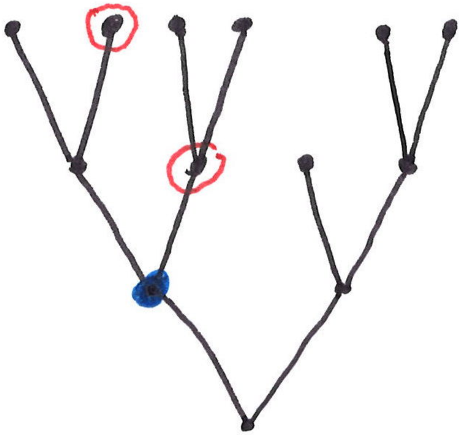
1. (Paths are consistent) For all $\eta \in 2^\omega$, $\{\varphi(x; a_{\eta|n}) : n < \omega\}$ is consistent.
2. (Weird inconsistency) If $\eta \perp \nu \in 2^{<\omega}$, $\eta \trianglerighteq (\eta \wedge \nu)^\frown\langle 0\rangle$ and $\nu = (\eta \wedge \nu)^\frown\langle 1\rangle$, then $\{\varphi(x; a_\eta), \varphi(x; a_\nu)\}$ is inconsistent.

We say T is NSOP$_1$ if no formula has SOP$_1$ modulo T.

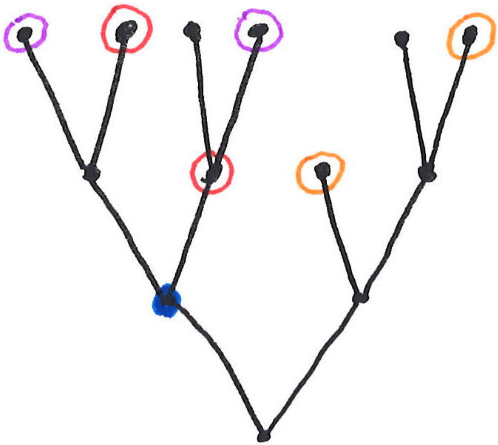
Consistency



Inconsistency



Agnostic



Arrays and trees

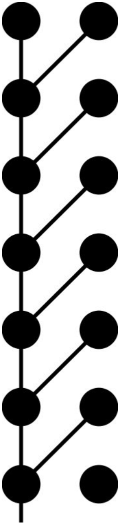
Lemma

(Chernikov-R.) The following are equivalent:

1. $\varphi(x; y)$ *has* SOP_1.
2. *There is an array* $(c_{i,j})_{i<\omega, j<2}$ *satisfying the following conditions:*
   2.1 $\{\varphi(x; c_{i,0}) : i < \omega\}$ *is consistent.*
   2.2 $\{\varphi(x; c_{i,1}) : i < \omega\}$ *is 2-inconsistent.*
   2.3 $c_{i,0} \equiv_{\overline{c}_{<i}} c_{1,i}$ *for all* i.

This is also true with 2 replaced by k, though the argument is a bit grungier.

The array



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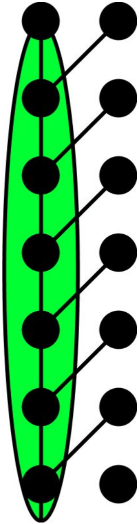

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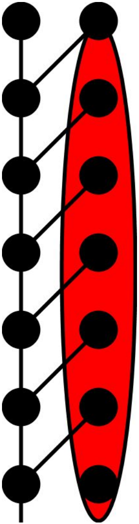

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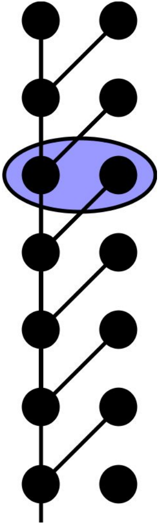

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From trees to arrays

- Assume $\varphi(x; y)$ has SOP_1 and, by compactness, we may assume that the witnessing parameters $(a_\eta)_{\eta \in 2^{<\kappa}}$ are indexed by a tree of height κ for some regular $\kappa > 2^{|T|}$.
- Inductively, we pick pairs (η_i, ν_i) in $2^{<\kappa}$ such that
  - The ν_i lie along a path and, if $i < j$ both η_j and ν_j extend $\nu_i^\frown\langle 0\rangle$.
  - If $i \leq j$, then $\eta_j = (\nu_i \wedge \eta_j)^\frown\langle 1\rangle$.
  - $a_{\eta_i} \equiv_{a_{\eta_{<i}} a_{\nu_{<i}}} a_{\nu_i}$ for all i.
- The induction step: having chosen η_i, ν_i, consider the sequence $(a_{\nu_i^\frown\langle 0^\alpha\rangle^\frown 1})_{\alpha<\kappa}$. By the choice of κ and pigeonhole, there must be $\alpha < \alpha'$ such that

  $$a_{\nu_i^\frown\langle 0^\alpha\rangle^\frown 1} \equiv_{a_{\eta_{<i}} a_{\nu_{<i}}} a_{\nu_i^\frown\langle 0^{\alpha'}\rangle^\frown 1}$$

  Put $\eta_{i+1} = \nu_i^\frown\langle 0^\alpha\rangle^\frown 1$ and $\nu_{i+1} = \nu_i^\frown\langle 0^{\alpha'}\rangle^\frown 1$.
- Then we have, by SOP_1
  - $\{\varphi(x; a_{\nu_i}) : i < \omega\}$ is consistent.
  - $\{\varphi(x; a_{\eta_i}) : i < \omega\}$ is 2-inconsistent,

  so we found the desired array.

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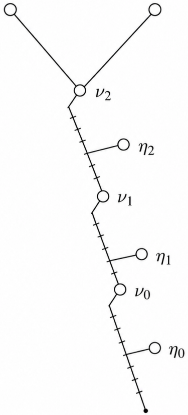

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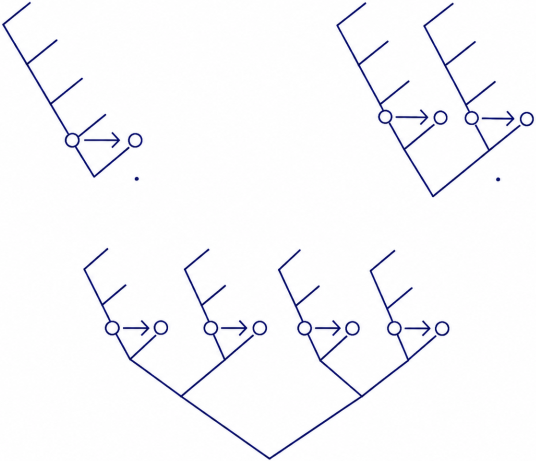
- Then we have, by SOP_1
  - $\{\varphi(x;a_{\nu_i}) : i<\omega\}$ is consistent.
  - $\{\varphi(x;a_{\eta_i}) : i<\omega\}$ is 2-inconsistent,

  so we found the desired array.

From arrays to trees

- Suppose we are given an array $(c_{i,j})_{i<\omega, j<2}$ satisfying the following conditions:
  1. $\{\varphi(x; c_{i,0}) : i < \omega\}$ is consistent.
  2. $\{\varphi(x; c_{i,1}) : i < \omega\}$ is 2-inconsistent.
  3. $c_{i,0} \equiv_{\bar{c}_{<i}} c_{i,1}$.
- Identify the array with part of a tree: $c_{i,0} = a_{0^i}$ and $c_{i,1} = a_{0^{i-1} \frown 1}$ for all $i > 0$.
- Then use condition (3) to build the rest of the nodes using automorphisms.

Building the tree



Array implies failure of Kim's Lemma

- Now suppose we have the array $(c_{i,0}, c_{i,1})_{i<\omega}$ satisfying
  1. $\{\varphi(x; c_{i,0}) : i < \omega\}$ is consistent.
  2. $\{\varphi(x; c_{i,1}) : i < \omega\}$ is 2-inconsistent.
  3. $c_{i,0} \equiv_{\overline{c}_{<i}} c_{i,1}$.
- We may assume that we are working in a Skolemization T^{Sk} of T, that we have stretched the array to $(c_{i,0}, c_{i,1})_{i<\omega+\omega}$, and that $(c_{i,0}, c_{1,1})_{i<\omega+1}$ is an indiscernible sequence.
- Let $M = \mathrm{Sk}(\overline{c}_{<\omega})$ and let $\mathcal{D}_j$ be a non-principal ultrafilter on M concentrating on $\{c_{i,j} : i < \omega\}$.
- By indiscernibility, we have $c_{\omega,j} \models \mathrm{Av}(\mathcal{D}_j, M)$ and since $c_{\omega,0} \equiv_{\overline{c}_{<\omega}} c_{\omega,1}$ and hence $c_{\omega,0} \equiv_M c_{\omega,1}$, we have $c_{\omega,0} \equiv_M c_{\omega,1}$.
- But, $(c_{\omega+i,j})$ is Morley over M in $\mathrm{Av}(\mathcal{D}_j, \mathbb{M})$ for $j = 0, 1$ (indexed in reverse), $\{\varphi(x; c_{\omega+i,0}) : i < \omega\}$ is consistent, and $\{\varphi(x; c_{\omega+1,1}) : i < \omega\}$ is inconsistent. This is a failure of Kim's Lemma for Kim-dividing.

Failure of Kim's Lemma implies array

- Suppose Kim's Lemma fails: so we have $\varphi(x; a)$ Kim-divides over M, witnessed by a Morley sequence $(a_i)_{i<\omega}$ in some global M-invariant type $r \supseteq \mathrm{tp}(a/M)$ and there is some global M-invariant type $q \supseteq \mathrm{tp}(a/M)$ such that, whenever $(a_i)_{i<\omega}$ is a Morley sequence over M in q, $\{\varphi(x; a_i) : i < \omega\}$ is consistent.
- Let $(c_{i,0}, c_{i,1})_{i\in\mathbb{Z}} \models (q \otimes r)^{\otimes\mathbb{Z}}|_M$.
- By choice of q and r, we have $\{\varphi(x; c_{i,0}) : i \in \mathbb{Z}\}$ consistent and $\{\varphi(x; c_{i,1}) : i \in \mathbb{Z}\}$ is k-inconsistent for some k.
- Moreover, for each i, $c_{i,0} \equiv_M c_{i,1}$ (since both have the same type as a over M) and

  $$(c_{>i,0}, c_{>i,1}) \models (q \otimes r)^{\otimes\omega}|_{Mc_{i,0},c_{i,1}}$$

  so, by M-invariance, $c_{i,0} \equiv_{M\bar{c}_{>i}} c_{i,1}$.
- Then $(c_{-i,0}, c_{-i,1})_{i<\omega}$ is the array we want.

Kim's Lemma for Kim-dividing

Theorem

(Kaplan-R.) The following are equivalent:

1. *T is NSOP$_1$*
2. *Kim's Lemma for Kim-dividing: for any model $M \models T$, $\varphi(x; a)$ Kim-divides over M if and only if $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent for every Morley sequence $(a_i)_{i<\omega}$ in a global M-invariant type extending $\mathrm{tp}(a/M)$.*

Thanks!