NSOP$_1$ theories

The evolution of independence

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nickramsey621.github.io/nsop1-tutorial.html

Outline

1. **The evolution of independence**
2. Trees, Kim's Lemma, and the generic scale
3. Symmetry

Model Theory

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Example

The language of groups is $L_{\mathrm{gp}} = \{e, \cdot, (-)^{-1}\}$. The axioms of a group are a theory in this language. A model of the theory of groups in L_{gp} is the same thing as a group.

Definable sets

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- If a is a tuple and A is a set of parameters, the *type* of a over A, denoted $\mathrm{tp}(a/A)$ is the collection of A-definable sets containing a. Tuples a and b from a (sufficiently saturated) model M have the same type over A if there is $\sigma \in \mathrm{Aut}(M/A)$ with $\sigma(a) = b$. In this case, we say a and b are *conjugates* over A.

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- This allows model theory to serve like a coarse algebraic geometry, treating structures like the free group (Sela), compact complex manifolds (Zilber, Moosa) and difference closed fields (Chatzidakis-Hrushovski).
- These geometric notions then play key roles in fine structural treatments of algebraic structures, e.g. reconstruction of groups on generic data, group classification (as in, e.g.. Hrushovski's proof of Manin-Mumford).

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- This led to the introduction of *Kim-independence*.

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  - What is a notion of independence?
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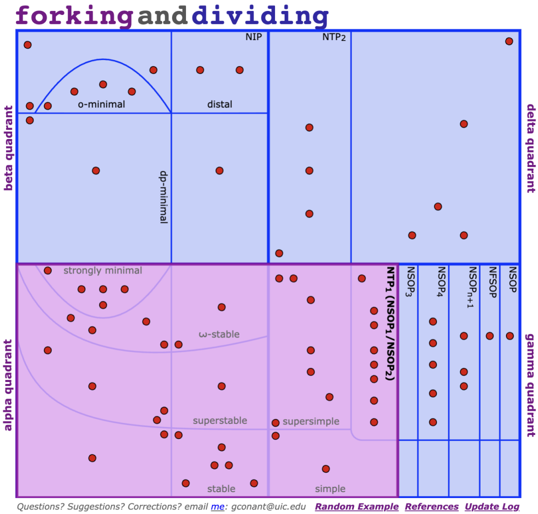
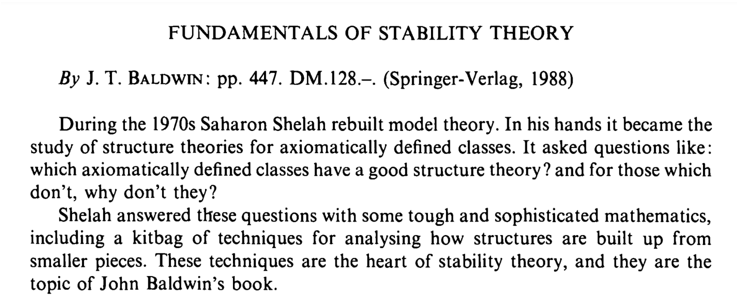
- Today we will try to answer these questions in reverse, starting with the most fundamental.

NSOP$_1$ theories



From Hodges' review of Baldwin's book



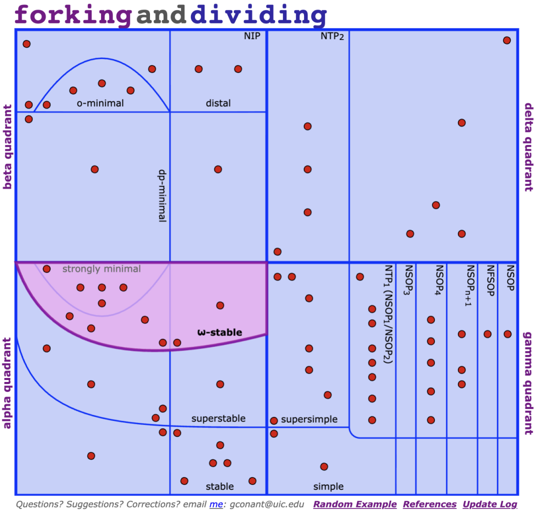
FUNDAMENTALS OF STABILITY THEORY

By J. T. BALDWIN: pp. 447. DM.128.–. (Springer-Verlag, 1988)

During the 1970s Saharon Shelah rebuilt model theory. In his hands it became the study of structure theories for axiomatically defined classes. It asked questions like: which axiomatically defined classes have a good structure theory? and for those which don't, why don't they?

Shelah answered these questions with some tough and sophisticated mathematics, including a kitbag of techniques for analysing how structures are built up from smaller pieces. These techniques are the heart of stability theory, and they are the topic of John Baldwin's book.

ω-stable theories



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- For a definable set X, $\mathrm{RM}(X) = 0$ iff X is finite, and $\mathrm{RM}(X) \geq \alpha + 1$ if there are infinitely many pairwise disjoint definable $Y_i \subseteq X$ with $\mathrm{RM}(Y_i) \geq \alpha$. Then $\mathrm{RM}(p) = \min_{X \in p} \mathrm{RM}(X)$. The ω-stable theories are exactly the theories where this rank is always ordinal-valued.
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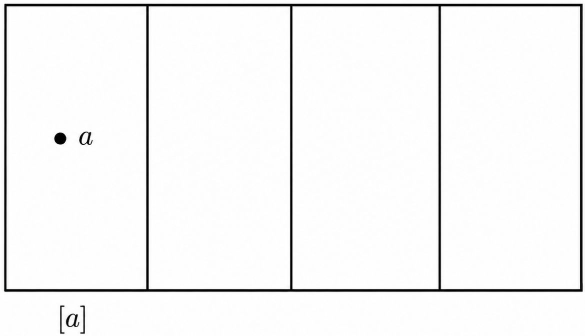
Morley rank: an example



$x = a$ has rank 0

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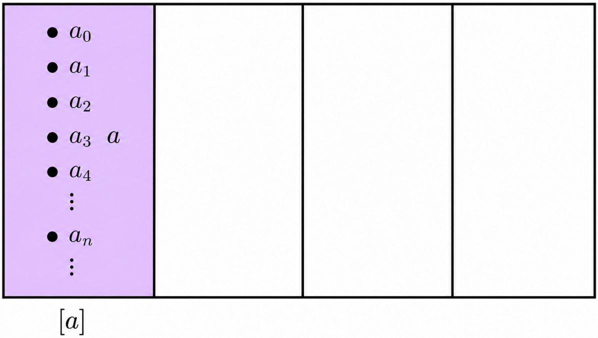
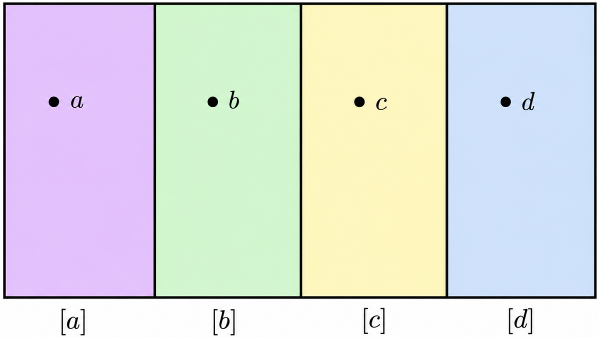


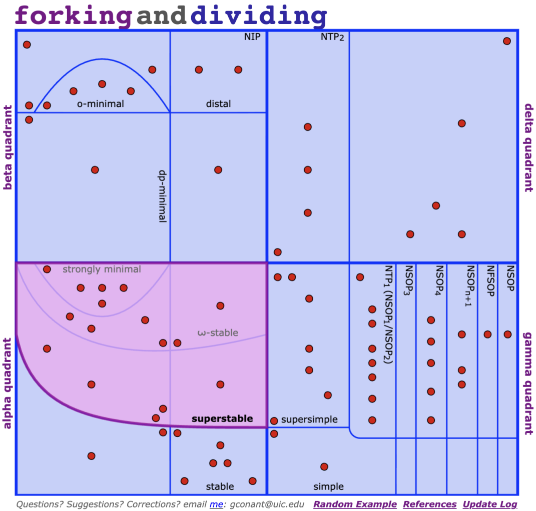
$E(x, a)$ has rank 1

Morley rank: an example



$x = x$ has rank 2

Superstable theories



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- This is a strictly broader class of theories: consider a language $L = \{P_\eta : \eta \in 2^{<\omega}\}$ consisting of unary predicates indexed by finite binary strings. Let T assert that $P_\emptyset$ holds of all elements and, for all η, P_η is partitioned into two non-empty pieces by $P_{\eta^\frown 0}$ and $P_{\eta^\frown 1}$. One easily sees $|S_1(\emptyset)| = 2^{\aleph_0}$ and $\mathrm{RM}(x = x) = \infty$.

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- The appropriate rank for superstable theories is R^∞. It is defined by: $R^\infty(X) = 0$ if and only if X is finite and $R^\infty(X) \geq \alpha + 1$ if and only if, for all cardinals λ, there are pairwise disjoint definable $Y_i \subseteq X$ such that $R^\infty(Y_i) \geq \alpha$. We define $R^\infty(p)$ to be $\min_{X \in p} R^\infty(X)$ for a type p.

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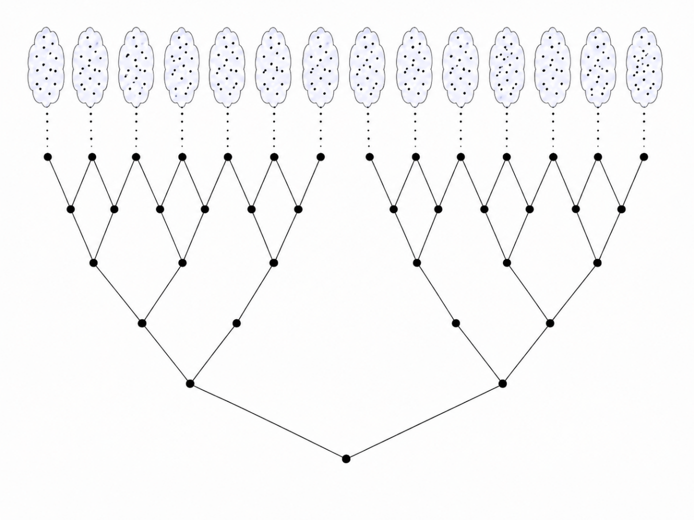

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- Using the rank, Shelah showed that, in time situations, one can decompose a structure into a collection of basic geometries and characterize the structure up to isomorphism by the dimensions attached to each one.

From rank to free extension

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- If $p \subseteq q$ and $R(q) = R(p)$, we think of q as a *free extension* of p and write $p \sqsubset q$.

Harnik Harrington's Fundamentals of Forking

To put it in other words, notice that any rank R defines a relation $\sqsubset$ between the complete types of T: $p \sqsubset q$ iff $p \subset q$ and $R(p) = R(q)$. Lascar's theorem says that the relation $\sqsubset$ defined by R is always the same, no matter which rank R satisfying I–IV we picked. If so, maybe that the important thing is not the rank but the relation $\sqsubset$ it defines! $\sqsubset$ is axiomatized via the rank R. Could we axiomatize it directly? Yes!

The Harnik-Harrington axioms

Axiom 0. If $p \sqsubset q$ then $p \subset q$ and if f is elementary then $p \sqsubset q$ iff $fp \sqsubset fq$

Axiom 1. If $p \subset q \subset r$, then:
(a) $p \sqsubset q \sqsubset r$ implies that $p \sqsubset r$,
(b) $p \sqsubset r$ implies that $p \sqsubset q$,
(c) $p \sqsubset r$ implies that $q \sqsubset r$.

Axiom 2. If $p \in S(A)$ and $A \subset B$, then $p \sqsubset q$ for some $q \in S(B)$.

Axiom $3_{\aleph_0}$. If $p \in S(A)$, then $p \upharpoonright A_0 \sqsubset p$ for some finite $A_0 \subset A$.

Axiom 4. For any p there is a cardinal λ such that there are at most λ mutually contradictory types q s.t. $p \sqsubset q$.

They also have **Axiom 3**: There is a cardinal κ such that if $p \in S(A)$, then $p \upharpoonright A \sqsubset p$ for some $A_0 \subseteq A$ with $|A| < \kappa$.

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- Then if $p = \mathrm{tp}(a/A)$ and $q \supseteq p$, we say q does not fork over A if q contains no formula that forks over A. This furnishes a notion of free extension of a type.
- Harnik and Harrington prove that, with respect to a fixed theory, any notion of free extension of types satisfying their axioms must be the notion of non-forking extension, *and* the theory must be stable.

Forking

- We say a formula $\varphi(x; a)$ *divides* over A if there are are infinitely many realizations $a_i \models \mathrm{tp}(a/A)$ and some k such that $\{\varphi(x; a_i) : i < \omega\}$ is k-inconsistent. One can demand demand that $(a_i)_{i<\omega}$ forms an A-indiscernible sequence and this is equivalent.

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  where each $\psi_i(x; c_i)$ divides over A.
- It's helpful to think about the sets defined by these formulas: declare sets defined by formulas that divide to be small. Then the sets defined by formulas that fork are exactly the sets in the ideal generated by the small sets.

Stable examples

- The theory of an equivalence relation with infinitely many infinite classes. If $p(x) \in S_1(A)$ is the type saying 'x is not equivalent to anything in A' then, for $B \supseteq A$, the non-forking extension is the unique type which says x is not equivalent to anything in B.

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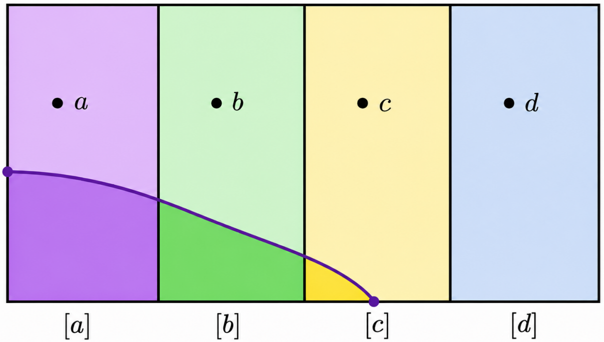
- The theory of an algebraically closed field of characteristic 0. Suppose $A \subseteq B$ are algebraically closed fields, and $p \in S_n(A)$ says x is in some irreducible variety V defined over A but not in any proper subvariety defined over A. The non-forking extension is the unique type which says x is in V but not in any proper subvariety defined over B.

The non-forking extension



$\mathrm{tp}(d/A)$ is the non-forking extension of $\mathrm{tp}(a/\emptyset)$

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$$A \underset{C}{\overset{f}{\downarrow}} B \iff \langle AC \rangle \cap \langle BC \rangle = \langle C \rangle.$$

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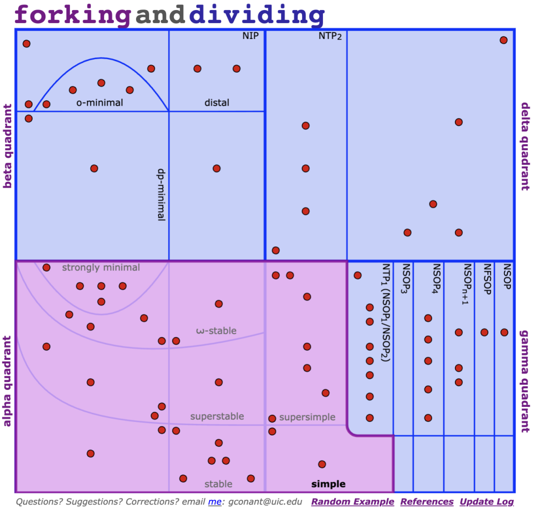
- The theory of a $\mathbb{Q}$-vector space. Non-forking is linear independence:

$$A \mathop{\smile\!\!\!\!\!\!|}^{f}_{C} B \iff \langle AC \rangle \cap \langle BC \rangle = \langle C \rangle.$$

- In an algebraically closed field, non-forking independence is algebraic independence.

Simple theories



Simple Theories

- The simple theories are the theories that do not have the *tree property*. We say $\varphi(x; y)$ has the tree property modulo T if there is a collection of tuples $(a_\eta)_{\eta \in \omega^{<\omega}}$ and there is some k such that
  - (Paths are consistent) For all paths $\eta \in \omega^\omega$, $\{\varphi(x; a_{\eta|n}) : n < \omega\}$ is consistent.
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    - (Siblings are inconsistent) For all $\eta \in \omega^{<\omega}$, $\{\varphi(x; a_{\eta^\frown\langle i \rangle}) : i < \omega\}$ is k-inconsistent.
- A guiding intution is that

$$\text{Simple} = \text{Stable} + \text{Random Noise}$$

Simple examples

- The random graph.
  - Not stable: for any set V of vertices and $X \subseteq V$, $\{R(x;v) : v \in X\} \cup \{\neg R(x;v) : v \in V \setminus X\}$ is consistent.
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  - Non-forking is disjointness: $A \mathop{\smile\!\!\!\!\!\!\,|}^{f}_{C} B \iff A \cap B \subseteq C$.
- An infinite dimensional $\mathbb{F}_p$-vector space with a non-degenerate symmetric bilinear form β.
  - Not stable: for any linearly independent set B and $X \subseteq B$, $\{\beta(x,v) = 0 : v \in X\} \cup \{\beta(x,v) \neq 0 : v \in B \setminus X\}$ is consistent.
  - Non-forking is linear independence: $A \mathop{\smile\!\!\!\!\!\!\,|}^{f}_{C} B \iff \langle AC \rangle \cap \langle BC \rangle = \langle C \rangle$

Simple examples

- The random graph.
  - Not stable: for any set V of vertices and $X \subseteq V$, $\{R(x; v) : v \in X\} \cup \{\neg R(x; v) : v \in V \setminus X\}$ is consistent.
  - Non-forking is disjointness: $A \mathop{\smile\!\!\!\!\!\!\;|}^{f}_{C} B \iff A \cap B \subseteq C$.
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- A pseudo-finite field.
  - Not stable: If the characteristic is $\neq 2$, $R(x, y) := (\exists z)[z^2 = x + y]$ defines a random graph on the field. [Duret]
  - Non-forking is algebraic independence.

From stationarity to the independence theorem

- Types over models (or over sets algebraically closed in the sense of T^{eq}) in stable theories are *stationary*: if $p \in S(M)$ and $A \supseteq M$, then if p_1 and p_2 are both complete types over A extending p which do not fork over M, then $p_1 = p_2$.

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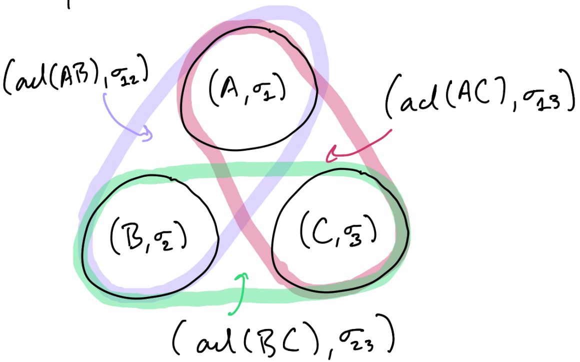
- In unstable simple theories, one has many non-forking extensions but you can put them together. Kim and Pillay proved **the Independence Theorem**: if T is a simple theory, $M \models T$, $A, B \supseteq M$ and $A \mathop{\smile\!\!\!\!\!|}^{f}_{M} B$, then for any $p \in S(M)$, if $q_1 \in S(A)$ and $q_2 \in S(B)$ are both non-forking extensions of p, then $q_1 \cup q_2$ is consistent and does not fork over M.

Independence theorem

$T = ACFA$

Theory of difference-closed fields



Kim-Pillay Theorem

The following theorem shows that non-forking is canonical in simple theories and gives a useful criterion for the class:

Theorem

(Kim-Pillay) Any theory in which there is an automorphism-invariant notion of independence which is symmetric, transitive, satisfies extension, local character, finite character, base monotonicity, and the independence theorem is simple. Moreover, any such notion must agree with non-forking independence.

The end of the road?

- Morley rank in ω-stable theories and R^∞ rank in superstable theories give rise to a notion of free extension/independence. Harnik and Harrington prove that actually any reasonable notion of independence satisfying stationarity must coincide with non-forking and the ambient theory must be stable.

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- Kim-Pillay generalizes this, showing that, dropping stationarity for the independence theorem, any reasonable notion of independence again must agree with non-forking and the ambient theory must be simple.
- Later, Kim seemed to show that this can't continue past simplicity. More precisely, he showed that if non-forking is symmetric or transitive, then the ambient theory must be simple. So in a non-simple theory, non-forking independence will not be a 'reasonable' notion of independence.

Non-simple examples

- Džamonja and Shelah introduced a theory called T^*_{feq}: the language consists of two sorts P and O and a ternary relation $E_x(y,z) \subseteq P \times O^2$. The theory T_{feq} says that, for each $p \in P$, E_p is an equivalence relation on O and T^*_{feq} is the model companion.

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- T^*_{feq} comes with a reasonable notion of independence:

$$A \overset{?}{\underset{C}{\downarrow}} B \iff P(A) \cap P(B) \subseteq P(C)$$

$$\text{and} \quad (\forall p \in P(C)) \left[O(A) \overset{f}{\underset{O(C)}{\downarrow}} O(B) \text{ in } (O, E_p) \right]$$

Non-simple examples

- Granger studied the two-sorted theory T_∞ of an infinite dimensional vector space V over an algebraically closed field K equipped with a non-degenerate alternating or symmetric bilinear form β.

Non-simple examples

- Granger studied the two-sorted theory T_∞ of an infinite dimensional vector space V over an algebraically closed field K equipped with a non-degenerate alternating or symmetric bilinear form β.
- T_∞ has a reasonable notion of independence (for definably closed A, B, C with $C \subseteq A \cap B$):

$$A \underset{C}{\overset{?}{\downarrow}} B \iff K(A) \underset{K(C)}{\overset{\mathrm{ACF}}{\downarrow}} K(B)$$

$$\text{and} \quad \langle A \rangle \cap \langle B \rangle = \langle C \rangle.$$

Non-simple examples

- Chatzidakis studied ω-free PAC fields. A field K is *pseudo-algebraically closed* (PAC) if every absolutely irreducible variety defined over K has a K-rational point. We say K is ω-free if $\mathrm{Gal}(K)$ is free profinite on infinitely many generators.

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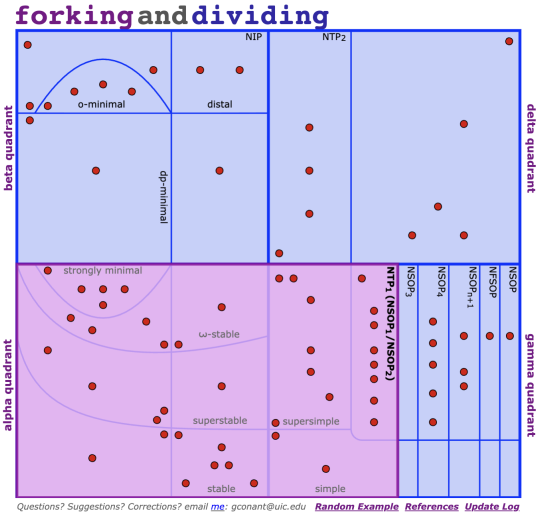
- She showed that ω-free PAC fields have a reasonable notion of independence, given by algebraic independence + 'non-forking independence in the absolute Galois group.'

Examples seen so far

stable	**simple**	**???**
non-forking	non-forking	???
ACF	ACFA	G-fields
SCF	PSF	PAC fields
Vector spaces	Classical geometries over finite fields	Classical geometries

$NSOP_1$ theories



NSOP_1 theories

The following definition is due to Džamonja and Shelah.

Definition

We say a formula $\varphi(x; y)$ has SOP_1 if there exists a collection of tuples $(a_\eta)_{\eta \in 2^{<\omega}}$ satisfying the following conditions:

1. (Paths are consistent) For all $\eta \in 2^\omega$, $\{\varphi(x; a_{\eta|n}) : n < \omega\}$ is consistent.

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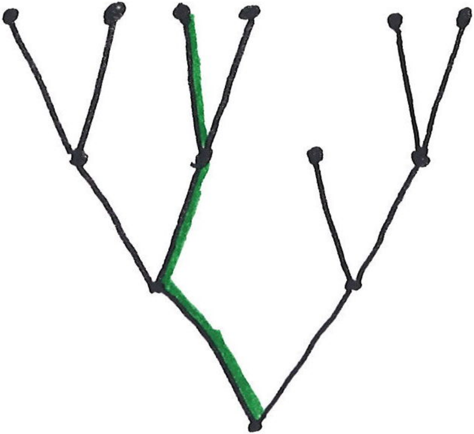
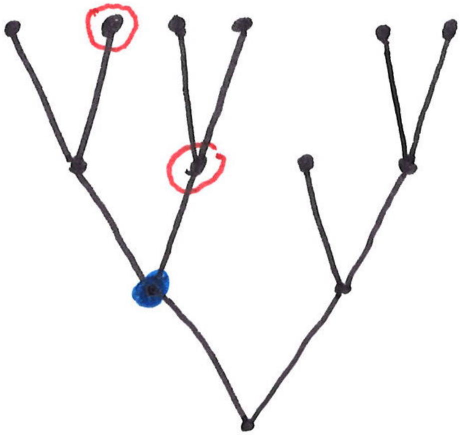
1. (Paths are consistent) For all $\eta \in 2^\omega$, $\{\varphi(x; a_{\eta|n}) : n < \omega\}$ is consistent.
2. (Weird inconsistency) If $\eta \perp \nu \in 2^{<\omega}$, $\eta \trianglerighteq (\eta \wedge \nu)^\frown\langle 0\rangle$ and $\nu = (\eta \wedge \nu)^\frown\langle 1\rangle$, then $\{\varphi(x; a_\eta), \varphi(x; a_\nu)\}$ is inconsistent.

We say T is NSOP_1 if no formula has SOP_1 modulo T.

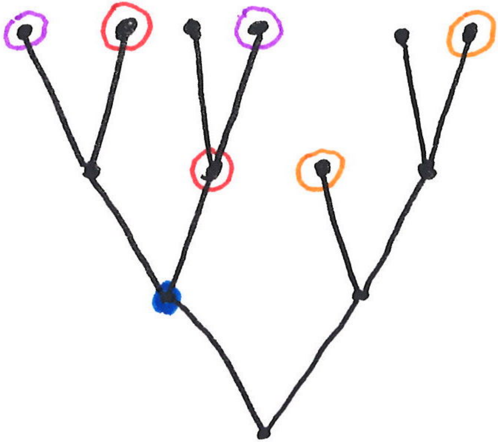
Consistency



Inconsistency



Agnostic



Kim-independence

Definition

Say that a formula $\varphi(x; a)$ *Kim-divides* over A if there is a k and a (sufficiently long) generic sequence $(a_i)_{i \in I}$ such that any k instances $\varphi(x; a_i)$ are inconsistent.

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Definition

Say that a is *Kim-independent* from b over C if a is contained in no set defined over b and C which Kim-divides over C. We denote this by $a \mathop{\smile\!\!\!\!\!\!|}\,^K_C\, b$.

The Main Theorem

Theorem

(Kaplan-R.) The following are equivalent:

1. *The theory T does not have the property SOP_1.*
2. $\mathop{\smile\hspace{-0.9em}\vert}^K$ *is symmetric: For all models M, $a \mathop{\smile\hspace{-0.9em}\vert}^K_M b$ if and only if $b \mathop{\smile\hspace{-0.9em}\vert}^K a$.*
3. $\mathop{\smile\hspace{-0.9em}\vert}^K$ *is transitive: For all models $M \prec N$, $a \mathop{\smile\hspace{-0.9em}\vert}^K_M N$ and $a \mathop{\smile\hspace{-0.9em}\vert}^K_N b$ implies $a \mathop{\smile\hspace{-0.9em}\vert}^K_M Nb$.*
4. $\mathop{\smile\hspace{-0.9em}\vert}^K$ *satisfies the independence theorem.*

A Kim-Pillay Theorem

The following theorem shows that non-forking is canonical in simple theories and gives a useful criterion for the class:

Theorem

(Chernikov-R., Kaplan-R.) Any theory in which there is an automorphism-invariant notion of independence which is symmetric, transitive, satisfies extension, finite character, witnessing, and the independence theorem does not have SOP_1. Moreover, any such notion must agree with Kim-independence.

Consequence 1: Classification

Using our Kim-Pillay theorem, the following examples have been shown to be NSOP_1 structures:

- **Combinatorics**: Generic structures (Kruckman-R.), generic projective planes (Conant-Kruckman), Steiner triple systems (Barbina-Casanovas), classical geometries over algebraically closed or pseudo-finite fields (Chernikov-R.).

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- **Combinatorics**: Generic structures (Kruckman-R.), generic projective planes (Conant-Kruckman), Steiner triple systems (Barbina-Casanovas), classical geometries over algebraically closed or pseudo-finite fields (Chernikov-R.).
- **Algebra**: PAC fields with free Galois group (Kaplan-R.), Frobenius fields (Kaplan-R.), Abelian varieties with a generic subgroup (d'Elbee), existentially closed exponential fields (Haykazyan-Kirby), Hilbert spaces with generic subset (Berenstein-Hyttinen-Villaveces), the random nil-2 group of exponent p for an odd prime p.

Consequence 2: Unification

- Algebraically or combinatorially defined notions of independence had been studied in PAC fields and vector spaces over algebraically closed fields.

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- Algebraically or combinatorially defined notions of independence had been studied in PAC fields and vector spaces over algebraically closed fields.
- These notions agree with Kim-independence, and so can be unified and explained on the basis of a general theory.

Consequence 3: Application

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1. **Fields** - In recent work with Chatzidakis, we construct measures on definable sets in PAC fields with free Galois group. We deduce the *definable amenability* of all definable groups in such fields, that is, all definable groups have invariant probability measures on their definable subgroups. This yields two interacting notions of genericity—one combinatorial, the other measure-theoretic—which has been applied by Bossut to classify them. .

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2. **Model theory** - Mutchnik's proof that $\mathrm{NSOP}_1 = \mathrm{NSOP}_2$ uses the theory of independence in an essential way.

Thanks!